PART 3: MULTIPLICATIVE OPERATIONS AND FRACTIONS

TEACHING CHILDREN MATHS

10 HOURS
Numeracy Academy

A team of writers from Bala Wande developed the Mathematics content of the Numeracy Academy drawing on the Bala Wande Thinking Maths modules, in consultation with Cally Khune of RED INK. The materials also draw on the Bala Wande Foundation Phase materials (Grades R to 3) were developed in consultation with a reference team of early Mathematics specialists.

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Bala Wande Grade R-3 Mathematics Programme Teacher Guides and Learner Activity Books. (2021, 2022, 2023).


Department of Basic Education. (2019). School Based Assessment (SBA). *Foundation Phase Mathematics SBA Exemplar Booklet. Grade 1 – 3*. Department Printers: Pretoria RSA.  


Welcome and Orientation

Welcome to the Numeracy Academy – this course aims to get you to start Thinking Maths!
The materials are divided into 4 parts. Each part should take you 10 hours to complete.
It is important that you go through the lessons in sequence as each lesson builds on the content
from the previous one.
We encourage you to be an active reader while engaging with each lesson.

Each lesson has video(s) that you need to watch by clicking watch now. If you are reading the
print version of the booklet you can use the QR code to access the video.

Each lesson also has a self assessment that you should complete. This will give you a chance to
recap on what you learned in the lesson.

Each lesson ends by providing you an opportunity to reflect on what you have learned in the
lesson. Take time to do this activity - it will help you consolidate and action the learning in your
classroom.

REFLECTION

• Reflect on your own school experience as a child. Did you have opportunities to play as a means
  of learning?
• Think about your own practice. Does your teaching provide opportunities for children to play? Or
  are your lessons mainly teacher-led? Write a personal goal in relation to enhancing the teaching
  and learning in your classroom by including playful experiences.
In order to gain the most from this course, please ensure that you watch the videos in full and that you complete each self-assessment. The assessments during the course are self-checks and the answers are given at the end of each lesson. As part of the final assessment of the module there will be two tests.

- Test 1 is taken after the completion of Part 1 and 2.
- Test 2 after the completion of Part 3 and 4.

- Each test lasts 1 hour and is in multiple-choice format.
- An online link for each test will be provided on the scheduled date.
- You will receive your results after clicking the submit button at the end of each test.
- If you fail the test you will be provided a second chance to take the test and a new date will be scheduled for this.

We hope you enjoy the course and find it beneficial!
PART 3:

MULTIPLICATIVE OPERATIONS AND FRACTIONS

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In Part 3 you will be introduced to multiplicative operations and fractions which form part of the content covered in the topic of Number, Operations and Relationships. Since, as you have learned about in Part 1 and Part 2, learners learn through play and interaction with other learners and adults, the activities presented in Part 3 show how the content of multiplicative operations and fractions can be taught in a play-based classroom, working interactively and through engaging with errors that arise in the learning process. In this module you will learn about the teaching of multiplication (introduced through repeated grouping activities) and division (introduced through sharing and grouping activities). You will also learn about the teaching of fractions, where it is shown that giving learners opportunities to work with a wide variety of wholes (continuous and discontinuous) is needed. Activities should be set in both in concrete and abstract problem contexts to enable the development of a solid concept of fractions as numbers.
In this lesson we begin our investigation of multiplicative reasoning. Multiplication is not an easy concept for learners to grasp, and you need to be sure that you have a sound understanding of the progression of learning in this topic and will be able to support learners to achieve the necessary conceptual understanding of the topic. We will start by looking at patterns in multiplication, because if learners recognise the patterns in Maths, then they find it easier to learn new concepts. We will also discuss times tables and consider ways to help learners learn these. Finally, we will look at language in Maths so that we can ensure learners use the appropriate vocabulary to verbalise their ideas and understanding.

What you will learn in this lesson
- Patterns in multiplication
- Times tables
- Maths language in multiplication

Patterns in multiplication

As we have discussed in previous lessons, patterns help us to make sense of the world around us. In the same way, if learners are able to identify multiplication patterns, they will find it easier to solve problems involving multiplication accurately and efficiently. You may not explicitly teach these patterns, but rather provide opportunities for learners to discover them for themselves. Ask probing questions that can guide this discovery. This will allow learners to construct their own understanding, rather than trying to remember rules or a recipe to follow.

We mentioned the importance of mathematical vocabulary in the previous section, and part of the language of multiplication includes the concept of multiples. We encounter multiples in both multiplication and division. Learners begin to discover and learn about multiples when they do repeated addition, or drill their multiplication tables. One would expect this to be a commonly known term, and yet it seems that it is not well understood at all. This may be because the term is not used sufficiently, and so we need to be sure to explain the term multiple to the learners and to use it as often as possible.

Some examples of multiples include:
- 2, 4, 6, 8, 10 are multiples of 2
- 7, 14, 21, 28 are multiples of 7
- 60 minutes in an hour are broken up into multiples of 5 on an analogue clock face

All numbers have an infinite number of multiples, while some pairs of numbers share certain multiples. To find common multiples, we simply write out some of the multiples of the given numbers and then look for those which are common to both. For example, numbers that are multiples of both 2 and 3 are called common multiples of 2 and 3.

Multiples of 2
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30

Multiples of 3
3 6 9 12 15 18 21 24 27 30

6, 12, 18, 24 and 30 are common multiples and 2 and 3.
ACTIVITY 1

More multiplication patterns

Watch the video “More multiplication patterns” (2:59 minutes), to see how the teacher uses multiplication patterns.

• Why do you think the teacher had blue and green marbles in the bags?
• What patterns do you notice in the video?

Commentary

In this video, we notice that the teacher has used blue and green marbles in the bags. She first asks the learners how many marbles there are altogether. The learners can see that there are 5 marbles in each bag so they can quickly work out that there are 20 marbles altogether. The learners do this by using repeated addition \((5 + 5 + 5 + 5 = 20)\) which we will discuss more in Lesson 2. The teacher then looks at the number of green and blue marbles, which encourages learners to break down 5 into \(3 + 2\) for each bag. Learners can therefore see a pattern in that all the bags have \(3 + 2 = 5\) marbles. They can also work out the number of marbles by counting in \(2s\) for the green marbles, \(3s\) for the blue marbles and \(5s\) for the total number of marbles. So, by using the blue and green marbles, there are three different calculations that can be worked out, even though there are only four bags of marbles in total.

Multiples of 2 pattern

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

Multiples of 3 pattern

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

Multiples of 5 pattern

\[
\begin{array}{cccccccccccccccc}
\end{array}
\]

Times tables

It is essential that learners know their times tables well, as so that they are able to recognise the relationships based on them. This will help learners to develop their confidence and accuracy in solving mathematical problems. For example, learners know that \(9 \times 7 = 63\) because \(10 \times 7 = 70\) and 7 less is 63. Based on their knowledge and understanding of patterns and relationships, learners are able to solve problems like these efficiently.
As part of identifying patterns and relationships, learners need to see the link between counting in multiples and “times tables”. Encourage learners to compare parts of the tables. So, for example, learners can look at $8 \times 4 = 32$ and $9 \times 4 = 36$. You can ask them what they notice, and what they think the next number in the sequence will be. Learners need to recognize that $9 \times 4$ is one group of 4 more than $8 \times 4$.

Learners therefore need a lot of practice with their multiplication tables. However, it is important that this practice should take place after the concept of multiplication has been thoroughly understood by the learners. Remember also that learners learn best through play, so the practicing of times tables should be made exciting and interesting for the learners by using games and other classroom activities.

**ACTIVITY 2**

How could you teach the four times table to learners?

- What can you do to make sure that learners construct their understanding rather than learning the times tables by rote?

**Commentary**

When learning their times tables, learners must build up each table so that they can see and understand where it comes from.

When learning the 4 times tables table, you could show the learners a toy car. Point out that you are holding one car, and that one car has four wheels. You can ‘park’ the car and write on the board $1 \times 4 = 4$, saying as you write “1 car with 4 wheels has 4 wheels in total”. You can then drive up another car, parking it next to the first car. You can write $2 \times 4 = 8$, saying “2 cars with 4 wheels each have 8 wheels altogether”.

You can carry on this way until you get to “12 cars with 4 wheels each have 48 wheels in total”. You can then go back to the beginning and discuss with the learners what they think will happen if we have zero cars. Let the learners see if they can write the number sentence by following the pattern that they have developed. The learners will see that if they have 0 cars with 4 wheels, then there will be 0 wheels in front of them.

**Maths language in multiplication**

The language of multiplication should be taught and used regularly in your lessons. Some of the terms associated with multiplication include multiply, times, groups of, sets of and multiples. Learners need to be comfortable using the correct terms as they verbalise their thought processes. It is important that learners know exactly what a multiplication problem is asking them to do. This means that, when they see the number sentence $4 \times 5 = \square$, they understand what the multiplication symbol means. Learners need to be able to recognise that the number sentence is actually saying “4 groups of 5”, which will help them to solve the problem. Make sure that learners know that it is not correct to say “times this number by 7”. The correct language is “multiply this number by 7”. We do however speak about “7 times 5” which means “multiply 5 by 7”.

As with addition and subtraction, when we introduce multiplication to our learners, we should use number stories that will lead to simple multiples. Make up stories yourself and call on the learners to make up some of their own too. If you allow them to make up their own questions from this early stage, they will develop their independence and ability to think creatively about mathematical situations.
ACTIVITY 3
Make up a story that would lead to a number sentence involving multiplication.

Commentary
As we discussed with addition and subtraction, you need to select items that interest the learners in your class. Then you can come up with problems, using simple sentences, that follow the same basic structure.

Encourage the learners to create their own multiplication stories, thinking about how to use the mathematical language that they have learnt. Make sure that you have modelled the appropriate use of the language so that the learners have a good example to follow.

Check your understanding: Multiple choice

1) Which sequence of numbers are multiples of 7?
   A) 2, 4, 6, 8, 10
   B) 7, 14, 21, 28, 35
   C) 1, 7, 17, 27, 37

2) I know that 4 x 8 = 32 because:
   A) I know that 4 x 10 = 40 and I can count backwards.
   B) I know that 4 x 5 = 20 and I can count on in 4s.
   C) Both A and B

3) The language of multiplication includes:
   A) Groups of, add, put together
   B) Groups of, sets of, times
   C) Groups of, sets of, add

4) When teaching multiplication, it is helpful to:
   A) Use problems in context
   B) Make learners learn algorithms off by heart
   C) Practice first, understand later

REFLECTION
• Reflect on your experience of learning times tables.
• Think about you felt when you were tested on your times tables.
• Write a personal goal in relation to the way you plan to approach times tables in your class.

Well done you have completed Lesson 1.
In this lesson, we will look into multiplication a bit more. We will start by considering how learners use concrete resources when solving multiplication problems. We will then move on to thinking about repeated addition, as this is how learners start solving multiplication problems, before talking about using arrays. This is the progression of learning that learners will follow, as they move from concrete apparatus to pictorial representations. We discussed the progression of learning when we learnt about addition and subtraction, and this progression is continued for multiplication and division.

What you will learn in this lesson

- Multiplication using resources
- Repeated Addition
- Multiplication using an array

**Multiplication using resources**

As we discussed when we looked at addition and subtraction, learners need to use concrete resources to help them visualise the process of multiplication. You can provide opportunities for learners to use counters or cubes, base ten blocks or a number line. Learners can physically arrange the resources into equal groups by using ten frames or circles as we saw in the video in Lesson 1.

Use multiplication stories to provide a variety of opportunities for learners to make equal groups. Let them talk amongst themselves about how they solve the problems. In order to think multiplicatively, learners must realise that they need to count equal groups. By physically grouping items into equal sets, they will become more efficient in their ability to reason and solve problems.

Once learners have used concrete resources to create a solid foundation of understanding, they can use arrays as a pictorial representation of multiplication. We will discuss arrays in more detail later in this lesson. Encourage learners to physically group concrete objects, or to draw circles around dots on an array. When counting items or dots, you should encourage learners to use skip counting so that that they solve problems more efficiently.

**ACTIVITY 1**

- Discuss the resources used to teach multiplication by teachers at your school.
- Which resources are most commonly used?
- Why are these the preferred resources?
- Which resources are not seen as valuable or helpful in the classroom?
- Why are these resources not highly regarded?
Commentary

It is a good idea to discuss teaching strategies and resources with your colleagues. Teachers all have their own teaching methods and styles, and we should generate conversations with other teachers to learn from their experiences. Just as we encourage learners to verbalise their solution strategies, so should teachers verbalise their teaching methods. By talking to others about what they are doing in the classroom, teachers will learn new ways of approaching challenging concepts with the learners. They will also discover which strategies are likely to be successful, and where potential problems may arise.

Repeated Addition

As we discussed in Lesson 1, when learners are given a multiplication problem to solve, they need to work out what they are being asked to do. Learners develop their understanding of multiplication through solving word problems that incorporate the appropriate language. For example, the problem might be:

**How many wheels are there on 11 bicycles?**

Learners could then use either concrete apparatus or drawings to help them solve the problem.

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||  ||  ||  ||  ||  ||  ||  ||  ||  ||  ||
```

It is important to encourage learners to verbalise their solution, and many learners may say “2 and 2 and 2 and 2 and 2 and 2 and 2 and 2 and 2 and 2 and 2 makes 22 wheels”. If they verbalise the solution in this way, you should then encourage learners to understand that they could use a multiple rather than repeated addition. They could ask themselves “How many groups of wheels are there?” Help learners to realise that there are “11 sets of 2 wheels”. This will help them to move away from repeated addition towards multiplication.

Once learners have developed their understanding of repeated addition, showing an ability to explain what they are doing to solve the problems, you can start encouraging them to calculate more efficiently. You can give learners a word problem where the repeated addition would be tedious. For example, “There are 12 tricycles. How many wheels altogether?” This will take learners a long time to draw or write out as a repeated addition problem.

```
|||    |||   |||    |||    |||    |||    |||    |||    |||    |||
```

3  +  3  +  3  +  3  +  3  +  3  +  3  +  3  +  3  +  3  +  3  = 36”

You could say to the learners “This is so much to write down. Can you think of a shorter way of writing this out?” You can encourage learners to recognize that there are 12 groups of 3 which can be written as 12 x 3 = 36. By doing this, learners will begin to think more efficient ways of solving multiplication problems. Using multiples is efficient and appropriate. Moving away from repeated addition is essential. The next activity would be appropriate at the Grade 3 level.

**Activity 2**

Solve the multiplication problem 63 x 5 without using a calculator.

- How did you solve the problem?
- Could you have found an easier way?
Commentary

As adults with an existing knowledge of multiplication, we might solve the problem $63 \times 5$ by using the column format (first calculation on the left). Learners who are not confident with the column format (or have not learned it) might choose to use repeated addition. Some alternatives are shown below.

It can be tricky to avoid errors when doing repeated addition. There are often a lot of numbers to add, and it can be easy to make mistakes. This leads learners to realise that multiplication is an easier solution method. There are even other ways that this calculation could be done – try to find a few!

Multiplication using an array

While it is a good idea to introduce times tables using concrete resources (as we discussed in Lesson 1), this is not always practical in terms of managing the resources in the classroom. An array is a great alternative to concrete resources, as you can place a cover over parts of the array and show any multiplication fact you want from $1 \times 1$ up to $10 \times 10$. In an array, we arrange dots (or other shapes) in columns and rows. We can then use the array to find the product of any multiplication question. For example, to multiply $3 \times 5$, we will create the following multiplication array.

<table>
<thead>
<tr>
<th>x</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

In this array, we can see that there are 3 rows and 5 columns. We can think of this as $5 + 5 + 5$ or as $3 \times 5$, and we can work out that there are 15 dots altogether. It is useful to write out your number sentences the same way each time to avoid confusion. In other words, the first number in the number sentence could be the number of rows, while the second number represents the number of columns. Ultimately learners realise that $3 \times 5$ is the same as $5 \times 3$. 
Arrays provide a visual representation of repeated addition or multiplication. A benefit of using arrays is that they help learners to see that the order of the numbers doesn’t matter in multiplication. Learners can turn the arrays around to see this clearly. As shown below learners will be able to see that $2 \times 6$ is the same amount as $6 \times 2$.

**ACTIVITY 3**

**Multiplication using array diagrams**

Watch the video “Multiplication using array diagrams” (2:57 minutes), to see how the teacher addresses the idea of an array.

- Why do you think the teacher uses a ten frame in the beginning of the video?
- How could you get learners to develop their understanding of multiplication by using arrays?

**Commentary**

In the video, the teacher uses a ten frame as an array so that the learners can easily rotate it to see that $2 \times 5$ is the same as $5 \times 2$. This is the commutative property of multiplication which we will discuss further in Lesson 3.

In order to develop learners’ understanding of multiplication, you could get them to show equal groups using an array. For example, you could ask them to use their array to show 4 groups of 3. The learners could then identify the number sentence $4 \times 3 = 12$. You could also get learners to break up the array in different ways. For example, a $4 \times 5$ array can be broken up into a $3 \times 5$ piece and a $1 \times 5$ piece. This will show learners that $4 \times 5$ is equal to $(3 \times 5) + (1 \times 5)$.

We know that learners move through three stages in terms of their understanding: concrete, representational, and finally abstract. You could use ten frames as a concrete array before moving on to representing the array in a drawing. Lastly, learners can solve multiplication problems in an abstract way, using only symbols. Remember that lots of practice in the concrete and pictorial stages to work efficiently and effectively in the abstract stage.
Check your understanding: True or false?

1. Concrete resources are a great way for learners to learn about equal groups.

2. Repeated addition is a quick and efficient method to solve problems.

3. An array is a visual representation of the laws of multiplication.

4. Based on their experience in adding and subtracting, learners can skip the concrete stage of learning in multiplication.

REFLECTION

• Reflect on your experience of using arrays in the classroom.
• Are you comfortable using an array to solve problems?
• Write a personal goal in relation to developing learners’ confidence in using arrays.

Well done you have completed Lesson 2.
In this lesson, we will investigate the laws of multiplication. It is necessary for you to understand the rules that multiplication follows so that you can teach these processes to the learners. The learners do not need to know the names of the laws, but they do need to recognise certain patterns that will help them to solve problems more efficiently. We will look at using base ten blocks in multiplication, considering how the concrete resources help learners to recognise and work with the laws of multiplication (even if they don’t know their names). Finally, we will discuss assessment, thinking about how to promote successful learning experiences for the learners.

What you will learn in this lesson

• The laws of multiplication
• Multiplication using base ten blocks
• Assessment

The laws of multiplication

Like addition and subtraction, multiplication has certain laws of operation. It is important for you to know these laws and to understand what they mean. The first law is that multiplication is **commutative**: This means that we can multiply a pair of numbers in either order, without changing the answer.

*For example, $7 \times 9 = 9 \times 7 = 63$.*

One of the strategies that we can teach our learners is to look out for “easier” ways of ordering in questions that involve multiplication of more than one pair of numbers. Another law is that multiplication is **associative**. This means that if we have to multiply a string of three or more numbers, we can do so by pairing them in any order that we choose.

*For example, $3 \times (2 \times 6) = (3 \times 2) \times 6$.*

A third law of multiplication is that it is **distributive** over addition and subtraction. This means that we can rearrange numbers in a number sentence to help us solve the problem in an easier way. For example, in the number sentence $2 \times (3 + 4) = \square$, we can solve the problem by adding $3 + 4$ and then multiplying by $2$ to get an answer of $14$. However, we could also multiply $3$ by $2$ to get $6$, and then multiply $4$ by $2$ get $8$. We could then add $6 + 8$ to get an answer of $14$. We could do the same thing with a number sentence that has subtraction instead. For example, in the number sentence $3 \times (10 - 7)$, we could solve the problem by working out $10 - 7 = 3$, and then multiplying by $3$ to get $9$. Alternatively, we could multiply $10$ by $3$ to get $30$, and then multiply $7$ by $3$ to get $21$. We could then take $21$ away from $30$, leaving us with an answer of $9$.

Multiplication has an **identity element**. This is the number which, when we multiply by, it has no effect on the multiplicand. The identity element of multiplication is the number $1$, since when you multiply a number by $1$ it does not change.

*For example, $1 \times 8 = 8 \times 1 = 8$*

Finally, it is also necessary to know that when a number is multiplied by zero, the answer will always be $0$. 

ACTIVITY 1

Test the laws of multiplication by working out the following examples:

- Calculate the following by pairing in different ways:
  
  $2 \times 17 \times 50 =$  
  
  Was there an order that was easier for you to do? If so, which one and why was it easier?

- Calculate the following:
  
  $13 \times 15 + 13 \times 5 =$  
  
  $13 \times 20 =$  
  
- What did you notice about the calculations?

Commentary

When you solve the problem $2 \times 17 \times 50$, you could work it out in the following ways:

$$
2 \times 17 = 34 \\
34 \times 50 = 1700
$$

This is actually quite tricky to work out, as $34 \times 50$ is not the easiest calculation to work out in your head. However, if you chose to solve the problem like this:

$$
50 \times 2 = 100 \\
17 \times 100 = 1700
$$

It is much easier to solve the problem mentally when you rearrange the numbers in a different order. $50 \times 2$ is a doubling fact that can be recalled quickly, and $17 \times 100$ is a simple calculation when you know about multiplying with tens and hundreds. These two methods of solving the problem show the associative law of multiplication.

When you solve the problem $13 \times 15 + 13 \times 5$, you need to work out $13 \times 15$ (which equals 195) first, and then $13 \times 5$ (which equals 65) before adding the two amounts together. So, you would add 195 + 65 to get an answer of 260.

To do this in another way, you could work out $13 \times 20$ - which you'll realise comes to the same answer of 260.

In this activity, you should have realised that the two calculations came to the same answer, despite the fact that you solved them in different ways. These calculations show the distributive law of multiplication. You need to give your learners opportunities to discover more efficient alternative ways of getting to solutions by combining numbers in different ways.

Multiplication using base ten blocks

When solving multiplication problems, it is useful to pack out base ten blocks as this can show the laws demonstrated in a concrete way. The base ten blocks are a physical representation of how we can rearrange quantities to help us solve problems more easily. Learners don’t need to learn the laws of multiplication by name, but they will understand the principles of these laws by seeing the blocks laid out and moved around to create new groups.

For example, using base ten blocks to solve a problem where a double-digit number is multiplied by a single digit number works well to show the distributive law. In solving this problem, the learners could attempt a variety of strategies, but they may find it easier to break up the double-digit quantity into tens and ones. They can group the tens together, making it easy to see how many there are altogether. They can then do the same with the ones. A physical representation of the tens and ones using base ten blocks will make it easier for the learners to multiply through a process of repeated addition. As you can see, this is exactly what we do in the distributive law, but it is far easier to understand it when we see it in a concrete form as opposed to an abstract number sentence.
ACTIVITY 2

Multiplicative reasoning 4 – using base ten blocks

Watch the video “Multiplicative reasoning 4 – using base ten blocks” (4:35 minutes), to see how the learners use base ten blocks to practice the distributive law.

- How do the learners pack out the base ten blocks?
- What do the learners need to do to help them solve the problem?
- How can you write out the process as number sentences?

Commentary

In the video, the learners are asked to solve the problem 4 x 12. They need to pack out the base ten blocks into tens and ones, recognising that 12 is made up of 10 + 2. They then need to group all the tens together and all the ones together (first image below). By grouping then tens together and the ones together, the learners can see that they have 4 groups of 10, and 4 groups of 2. This helps them to break up 4 x 12 into 4 x 10 + 4 x 2, which simplifies the calculation process for them (second image below).

```
4 x 12 = 4 x (10 + 2)
= 4 x 10 + 4 x 2
= 40 + 8
= 48
```

Assessment

When assessing learners’ learning of multiplication, you need to provide a number of opportunities for them to show you what they understand about the process of multiplying. You will then be able to collect evidence through observation and engagement with the learners. Once you have gathered the information you need from working with the learners, you can evaluate the evidence and make decisions about the learners' levels of understanding. Following this, it is important for you to record your findings, so that you can refer back to these at a different time. As we have discussed previously, it is essential that you use information gathered from assessing learners to plan for future lessons, and to redirect the teaching and learning process where necessary.

It is difficult to accurately assess too many learners at one time. This is why some teachers revert to summative assessment, as it seems easier to just mark a written task and see how many problems the learners were able to get correct. However, summative assessment does not take into consideration the learners' progression of learning or their context. This means that, sometimes summative assessment might not be as accurate or informative as you would like.
Therefore, when setting up formal assessment tasks with your learners, try to work with a group of learners at a time so that you have the opportunity to interact with them. It may take a few days for you to complete your assessments, but you will end up with a far deeper understanding of the learners’ knowledge and skills if you have been able to ask them questions and to listen to their justifications.

**ACTIVITY 3**

Plan and teach an activity where you can assess your learners’ understanding of multiplication.

- Design an observation sheet that will focus your interaction with the learners.
- Think about how you will record your observations of the learners.
- How will your observations direct your follow-up lessons?

**Commentary**

Observations about the learners in your class should be factual, accurate and objective (exactly what you see and hear). It is important that you do not allow your own opinions, biases or assumptions to influence what you observe, as this may affect your observation in a negative way. To make sure that observations are factual, accurate and objective consider the following ideas:

Write down the facts. What is the learner doing? What are they saying?

- “Julie is sitting at the table, doing a puzzle. She has placed five pieces together. She is holding one piece and trying to find where to place it.”

Think about whether someone else observing would describe the learner’s behaviour in the same way. For example, two observers might interpret the behaviour differently:

- “Julie can’t finish the puzzle. The remaining pieces are just lying on the table.”
- “Julie has sorted the remaining puzzle pieces according to their colour so that she can join the pieces together that are the same colour.”

Do not make assumptions. Do not write what you think is happening about what the learner can or cannot do, or how you think the learner is feeling. You may miss things that do not fit with your opinion.

- “Julie can’t do the puzzle. She is confused and frustrated and angry and is trying to force the piece to fit.”

Describe what you see and hear in as much detail as possible. This will help you to identify the learner’s achievements and needs as accurately as possible.

- “Anathi is matching number symbols and number names. He puts all the symbols 1 to 8 in order on the table. Then he puts the number names from one to six next to the matching symbol. He holds the number cards seven and eight in his hand and starts to cry, saying ‘I don’t want to do this’.”

Be aware of your own biases. You may have different expectations for girls and boys, or you may assume that a learner with barriers to learning is at a disadvantage. It is important not to jump to conclusions and give each learner the same consideration.

Teachers can also create an observation checklist. This is used to indicate whether a learner is competent, partially competent or not yet competent with a particular concept/skill.

Checklists provide columns into which you can enter information about the achievement (fully/partially/not at all) in relation to given criteria. The first exemplar checklist can be used after teaching a specific topic or at the end of a series of lessons to record more than one topic.
<table>
<thead>
<tr>
<th>Mathematics: GRADE 1: TERM 3: Checklist</th>
</tr>
</thead>
<tbody>
<tr>
<td>Numbers, Operations &amp; Relationships</td>
</tr>
<tr>
<td>- achieved</td>
</tr>
<tr>
<td>Estimation and counting reliably to 40 by using the strategy of grouping</td>
</tr>
<tr>
<td>Counts forwards and backwards in Ones from any number between 0 and 80</td>
</tr>
<tr>
<td>Counts forwards in multiples of 10s, 2s and 5s between 0 &amp; 80</td>
</tr>
<tr>
<td>Recognises, identifies, reads and writes number names 1 to 10</td>
</tr>
<tr>
<td>Reads number symbols 1 to 80</td>
</tr>
<tr>
<td>Recognises the place value of numbers 11 to 19</td>
</tr>
<tr>
<td>Decomposes two-digit numbers into ten ones.</td>
</tr>
<tr>
<td>Copies, extends and describes simple number sequences</td>
</tr>
<tr>
<td>In 1s, 10s, 5s, 2s to at least 80</td>
</tr>
<tr>
<td>Recognises and draws line of symmetry in 2-D geometric and non-geometric shapes</td>
</tr>
<tr>
<td>Estimates, measures, compares, orders and records length using non-standard measures</td>
</tr>
<tr>
<td>Uses language to talk about the comparison</td>
</tr>
<tr>
<td>Answers questions about data in pictographs</td>
</tr>
</tbody>
</table>

Date

Names of learners

1
2
3
4
5
6
7
8
9
10
11
12

SBA (2019), Grade 1 Checklist.

The second exemplar checklist is designed as an observation tool for the assessment of learners’ progress in the learning of numbers (0-5).

Observe learners to assess their ability to match, sort, order and compare numbers up to 5

Mark: 7

Criteria checklist: correct/incorrect/almost

- Able to match counters to objects
- Able to sort counters onto a five frame
- Able to count a number of objects up to 5
- Able to compare numbers to say which one is more (greater) than another
- Able to compare numbers to say which one is less (smaller) than another
- Able to recognise the number symbols 0 to 5
- Able to write the number symbols 0 to 5

This rubric is designed to make a record of levels of ability in a Grade 3 class in relation to levels of understanding of the topic of symmetry.

Space and shape: Assess the learners’ ability to recognise and work with symmetry.

<table>
<thead>
<tr>
<th>Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
<tr>
<td>5</td>
</tr>
<tr>
<td>6</td>
</tr>
<tr>
<td>7</td>
</tr>
</tbody>
</table>

- Able to identify the shape but unable to recognise when a shape is symmetrical
- Able to recognise when a shape is symmetrical but cannot show the line of symmetry
- Able to recognise when a shape is symmetrical and show one line of symmetry
- Able to recognise when a shape is symmetrical and can show more than one line of symmetry
- Able to draw a symmetrical shape with one line of symmetry
- Able to draw a symmetrical shape with more than one line of symmetry
- Able to draw a symmetrical shape or pattern and describe symmetry in patterns where more than one symmetrical shape is present
You can design your own checklists or rubrics to record observations. Take time to think carefully about the criteria you develop for the instruments, so they enable you to distinguish between levels of understanding of the topic under observation.

**Check your understanding: Multiple Choice**

1) Which number sentence below proves the commutative law of multiplication?
   - A) $9 \times 2 \times 3 = 9 \times 6$
   - B) $9 \times 3 = 3 \times 9$
   - C) $9 \times (3 + 2) = 9 \times 3 + 9 \times 2$

2) Which number sentence below proves the associative law of multiplication?
   - A) $4 \times 5 \times 2 = 4 \times 10$
   - B) $4 \times 7 = 7 \times 4$
   - C) $4 \times (3 + 5) = 4 \times 3 + 4 \times 5$

3) Observations for assessment should be:
   - A) Brief
   - B) Subjective
   - C) Factual

4) Teacher bias is:
   - A) A myth
   - B) A gut feeling based on experience
   - C) An assumption to be avoided

**REFLECTION**
- Think about how you assess learners in your class.
- Do you make assumptions about what you think they know?
- Where do these assumptions come from?
- What can you do to prevent these assumptions influencing your assessments?

**Well done you have completed Lesson 3.**
In this lesson, we will discuss the concepts of doubling and halving. These processes form an important part of learners' number fact knowledge, and a sound understanding of doubling and halving will help them to solve problems accurately and efficiently. We also discuss the mathematical language of division, as we have done for addition, subtraction and multiplication. Finally, we investigate patterns in division, and how these help learners to simplify the way they solve division problems.

**What you will learn in this lesson**
- Halving and doubling
- Maths language in division
- Patterns in division

**Halving and doubling**

When learners understand halving and doubling, they are able to solve certain problems mentally, using their knowledge of number facts. It is therefore essential that you find practical ways to teach the concepts, giving learners multiple opportunities to work with concrete apparatus in playful ways.

![Halving and Doubling Diagram](image-url)

**Halving**

- Means that a number is divided into 2 equal parts. Each half is the same amount.

**Halving games**
- Halving paper shapes
- Halving counter / pebbles
- Fast cards - turn over even number cards and halve as fast as possible

**Doubling**

- Means two of the same amount added together

**Doubling games**
- Snap
- Dominoes
- Dice Roll - roll dice and double the number as fast as possible
- Doubles memory game

An array allows learners to see the laws of multiplication and division concretely. They can physically handle the apparatus. For example, if learners are given the problem 4 x 5, they can create an array of 4 rows and 5 columns. They can then arrange the counters in different ways to help them solve the problem. If the learners rearrange the rows into 2 rows of five, they can see the problem has changed to 2 x 10. The two displays of marbles on the next page illustrate these two different ways in which the arrays can be arranged.
This is interesting because if you look at the two number sentences found here, you will notice a specific pattern. If you halve the first number in the number sentence, and double the second number, you will get the same answer as you would have from the original number sentence.

4 × 5 = 20
halve    double
2 × 10 = 20

This is a useful strategy for solving multiplication problems, but it is important for learners to understand the strategy. They need to realise that they should only use halving and doubling to solve a problem, if it is the most efficient way. For example, to try and solve 17 x 15 by halving and doubling you would do the following:

17 × 15 = 20
halve    double
8,5 × 30 = 20

This is not an easier way to solve the problem. There are other strategies that could be used more easily.

**ACTIVITY 1**

Watch the video "Halving" (3:26 minutes), to see how the teacher uses base ten blocks to help the learners halve.

- How would you introduce halving?
- Do you think the base ten blocks are helpful in getting learners to develop their understanding?
- Would you use a similar method to teach doubling?
Commentary

In this video, the teacher demonstrates halving the number 68 using base ten blocks. She packs out 6 tens and 8 ones so the learners can clearly see how the number is constructed. This then makes it easier for the learners halve, as they can halve the tens and the ones separately. If they have 6 tens, they are able to easily halve that and say that there will be 3 tens. They can also halve 8 ones, recognising that they will be left with 4 ones. When they add 3 tens and 4 ones, they will see that half of 68 is 34. This is one way to do halving. Try to think about other ways and other contexts that lend themselves to solution of problems using doubling and halving.

Maths language in division

As with all of the other operations, learners need to practice using the maths terminology to express their understanding of division. Learners need to be able to use the language to talk about the way that they solve problems. In the beginning, the concept of division can be explored without the learners even realising that they are dividing. By giving them real life problems, the learners can develop their understanding of the concept in a practical way. For example, the teacher could give learners the following problem:

| I have 15 hats. |
| The hats must go on 3 shelves. |
| How many hats will go on each shelf? |

The learners can use concrete resources to solve the problem, and they need to be able to verbalise their thinking. They are likely to use social language initially, so it is important for teachers to model the appropriate language to help learners develop their conceptual understanding. As learners copy the teacher, and practise using the language, they can share their methods and learn from each other. Once they are comfortable doing this, the teacher can then introduce the language of division. She can teach the learners the words divide, share, group and remainder, giving them a variety of word problems to demonstrate the meaning of these words. As learners develop their confidence in their ability to solve division problems, teachers can then introduce the symbols of division.

ACTIVITY 2

- Write about some activities that will encourage the use of the new vocabulary of division - division, divide, group, share, grouping and sharing.
- When would you do these activities?

Commentary

Learners will develop a better understanding of the appropriate mathematical terminology if they are given many opportunities to practice using the language in context. You need to give them many different grouping and sharing stories that lead to problems they will solve. The word divide or division only is used from Grade 3 onwards. However, it can be challenging for teachers to find time to focus on this during their planned maths lessons. We must remember that we do not need to only use division stories during scheduled division tasks. We can give learners Mental Maths problems that encourage them to use the language of division, even if we are focusing on a different topic in the actual Maths lesson. We could also prepare ‘Pick up’ activities which are available for learners to pick up and work on independently when they have completed set tasks. In this way, learners will be able to practice their division stories regularly and maintain their understanding of the terminology and the problems.
Patterns in division

Knowledge of place value and number properties such as divisibility can help learners to carry out a range of calculations mentally. When learners know their multiplication and division facts, they are able to use this knowledge to solve many problems in their heads simply by understanding the divisibility rules. Divisibility rules are patterns of division (some are simple, some more complex) that allow us to check whether certain numbers are divisible by other numbers (with no remainder) without actually dividing. Divisibility rules therefore enable us to check quickly whether a number is a factor of another number.

**Divisibility rules**

- **2**
  - Even numbers are divisible by 2
- **4**
  - The last two numbers must be divisible by 4
- **5**
  - Numbers that end in 5 or 0 are divisible by 5
- **6**
  - The number must be divisible by 2 and 3
- **8**
  - The last 3 numbers must be divisible by 8
- **9**
  - The sum of the digits must be divisible by 9
- **10**
  - Numbers that end in 0 are divisible by 10
- **3**
  - The sum of the digits must be divisible by 3

**ACTIVITY 3**

Solve the problem and then answer the questions below:

* I have 20 biscuits to pack with 5 biscuits in each box. How many boxes will I need?
  - Will this divide without leaving a remainder?
  - How did you decide?
  - What other numbers will divide into 20 without leaving a remainder?
  - How did you decide?

**Commentary**

When solving a problem like this, learners need to make sure that they understand the key information. Encourage learners to look carefully at the number facts in the problem, and to verbalise in their own words what they think they need to do to solve the problem. Watch learners as they solve the problem and note whether they use their knowledge of multiplication facts to help them. Do learners remember their 4 and 5 times tables? They could use the number fact $4 \times 5 = 20$ to help them work out that $20 \div 5 = 4$. Talk to the learners about their responses and their reasons. Give them opportunities to explain ideas such as “2 will work because 2 tens is 20 and 20 is an even number so it will divide equally by 2”.
Check your understanding: True or false?

1) Halving and doubling can be used as an efficient solution strategy for multiplication problems.

2) Maths language can only be developed in scheduled Mathematics lessons.

3) Patterns only fall into the curriculum topic, Patterns, functions and algebra.

4) Learners should have a sound understanding of number facts to help them solve problems.

REFLECTION

• Reflect on your experience of patterns in Mathematics.
• Do you use your recognition of patterns in Maths to help you solve problems?
• How can you encourage learners to identify and use patterns in their own Maths lessons?

Well done you have completed Lesson 4.
In this lesson, we will look at the laws of division and what learners need to know about dividing. As they develop their understanding of division, learners need to recognise the difference between sharing and grouping. We will investigate these differences and discuss examples of problems for both sharing and grouping.

What you will learn in this lesson
- The laws of division
- Grouping
- Sharing

The laws of division
We will now look at the laws of division – and what we find is the laws that applied to multiplication do not apply to division. Unlike multiplication, division is not commutative: This means that we cannot change the order of the numbers in a division number sentence without changing the answer. This can be seen in the following pair of number sentences:

\[35 \div 7 = 5 \quad \text{and} \quad 35 \div 5 = 7\]

Division is also not associative. This means that we cannot pair numbers in different orders when working with three or more numbers. This is shown in the following pair of examples:

\[(36 \div 6) \div 2 = 6 \div 2 = 3 \quad \text{and} \quad 36 \div (6 \div 2) = 36 \div 3 = 12\]

Unlike multiplication, division is not distributive over addition and subtraction. This means that we cannot rearrange numbers in a number sentence to help us solve the problem in a different way. For example, in the number sentence \[35 \div (2 + 5) = \square\], we solve the problem by adding 2 + 5 and then dividing 35 by 7 to get an answer of 5. However, if we were to try solve the problem by saying \[35 \div 2 + 35 \div 5 = \square\] we would end up with \[17.5 + 7 = 24.5\] which is not the correct answer.

When you divide 0 by another number the answer is always 0. For example: \[0 \div 2 = 0\]. This means 0 sweets shared equally amongst 2 learners, resulting in each learner getting 0 sweets. When you divide by 1, the answer is the same as the number you were dividing. \[2 \div 1 = 2\]. For example, if two sweets are divided by one learner, then the one learner would get both sweets.

Division by 0 is not defined. This is often misunderstood in mathematics and needs to be dealt with carefully so that learners understand it better and it does not lead to problems. If we consider the problem: You have 2 sweets but no learners to divide them among you cannot do anything. We realise that \[2 \div 0\] is not possible. You cannot divide by 0.

ACTIVITY 1
Refer back to our discussions on the laws of addition, subtraction, multiplication and division.
- What are the similarities that you notice between the laws relating to the four operations?
- What are the differences between the laws relating to the four operations?
Commentary

It is essential and useful for teachers to identify the similarities and differences between the four operations. This ability to recognise patterns in mathematics helps us to simplify our processes and solidify our conceptual understanding.

- The commutative and associative laws apply to single operations and they hold for addition and multiplication. They do not apply to subtraction and division.
- The distributive law is a little more complex in that it involves more than one operation. Understanding the distributivity of multiplication over addition and subtraction is the key to solving multiplication and division problems. Multiplication is distributive over addition and subtraction from both the left and the right, as can be seen by the following number sentences:

  Left: \[2 \times (3 + 4) = (2 \times 3) + (2 \times 4)\]
  Right: \[(2 + 3) \times 4 = (2 \times 4) + (3 \times 4)\]

  Left: \[4 \times (3 - 2) = (4 \times 3) - (4 \times 2)\]
  Right: \[(4 - 3) \times 2 = (4 \times 2) - (3 \times 2)\]

- Division on the other hand, can only be distributed over addition from the right, in the sense that \[(80 + 20) \div 8 = 80 \div 8 + 20 \div 8\].

This is quite a technical discussion, but it is useful for teachers to have a higher level of understanding of the operations and the relationships between them. This will help you to teach learners at the level they are at and answer their questions meaningfully and appropriately. Come back to this discussion later to consolidate this knowledge if necessary. We now discuss the activities of grouping appropriate to the early grades.

There are two ways of conceptualising division. They are known as grouping and sharing. When we give learners word problems using real-life contexts, they learn that these can be solved in either a grouping or a sharing way. This means that, whether we think of 10 divided by 2 in a grouping or a sharing way, the answer will be the same, and only the strategy is different. We must be sure to explain both division strategies to our learners, and so we will aim to discover more about these two ways of dividing by looking at examples of each.

### Grouping

Read the following problem and think about how you would solve it.

<table>
<thead>
<tr>
<th>I have 90 sweets.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I sell them at the cake sale in bags with 6 sweets per bag.</td>
</tr>
<tr>
<td>How many bags can I make?</td>
</tr>
</tbody>
</table>

The number sentence for this problem is \[90 \div 6\] bags.

But, to work it out, I need to put six sweets in each bag. I will keep doing this until all the sweets are used up and then I count how many bags I can make.
This is known as **grouping division**. This division strategy is often neglected, though it occurs often in real situations.

In *grouping*, you have to make groups of objects first. You find the solution by counting the number of groups. This can be done by using repeated subtraction – as an introductory strategy and by using multiples later on.

**ACTIVITY 2**

Look at the illustrations below and explain the difference between them as solutions to the number sentence $10 \div 2 = \square$.

Two children can get 5 lollipops each

There are 5 bags with 2 lollipops in each bag

**Commentary**

When solving these problems, it is apparent that the number sentence $10 \div 2 = \square$ is a simple one, that results in the answer 5. However, when we look at the illustrations, we notice that in the first one, there are two learners, whereas in the second one there are 5 packets. This tells us that the number of ‘groups’ is different in each solution. In the first solution, 2 groups of 5 sweets in each group have been made (so 2 learners could get sweets). In the second solution, 5 groups with 2 sweets in each group (bag) (so 5 bags of sweets have been made).
Sharing

In sharing, you can’t make groups of objects because you don’t know how many objects should be in one group because you first have to share out the items to find out how they are shared. When you share, you have to distribute the objects one by one (or using pairs/other groups, when you are ready to share more efficiently and you have a lot of items to share).

Read the following problem and think about how you would solve it.

I share 65 sweets among 7 learners.
How many sweets will each learner get?

The number sentence for this problem is $65 \div 7$ sweets. The way in which we do it is we share out the sweets, until they are all given out, and then we find out how many sweets each learner got. This is known as sharing division. Many division word problems are phrased in this way.

When the sweets are shared out in this problem, each learner ends up with 7 sweets and there are 2 sweets left over.

ACTIVITY 3

Watch the video “Division (sharing and grouping)” (5:14 minutes), to see the difference between grouping and sharing.

• What do you notice about the way the teacher demonstrates solving the problem $45 \div 5$?
• What do you notice about the way that the teacher demonstrates solving the problem $27 \div 3$?
• What did you notice about the language the teacher used as she spoke about these problems?
Commentary

It is clear from the language that the teacher uses that there is a distinct difference between the two problems identified in this activity. The context of the problem determines the kind of division strategy that must be used. When you look at the number sentences, they look just like division problems:

45 ÷ 5 = □
27 ÷ 3 = □

However, as you listen to the terminology used by the teacher you can tell that the first problem is a sharing problem. The teacher shares out the sweets between 5 learners so that the sweets are divided equally. This means that you know the total number of sweets and you know how many learners (or groups) you have, and you need to work out how many sweets go into each group. When you compare this to the second problem, it is clear that the second problem is a grouping problem. The teacher tells the learners that there are 27 flowers in total, and that the flowers are put into groups of 3. This means that they know the total number of flowers and the number of flowers that go into each group, and they then need to work out how many groups of flowers they will have in the end.

Check your understanding: Multiple Choice

1) Division is:
   A) Commutative  
   B) Associative  
   C) None of the above

2) 7 ÷ 0 =
   A) 7
   B) Not possible
   C) 0

3) In grouping problems, we know:
   A) The total number of objects, and how many groups there are.
   B) The total number of objects, and how many are in each group.
   C) How many groups there are and how many are in each group.

4) In sharing problems, we know:
   A) The total number of objects, and how many groups there are.
   B) The total number of objects, and how many are in each group.
   C) How many groups there are and how many are in each group.

REFLECTION

- Reflect on your own understanding of grouping and sharing.
- Did you understand the difference between grouping and sharing when you were at school?
- Write a personal goal in relation to how you will help learners to grasp the difference between these two types of problems.

Well done you have completed Lesson 5.
In this lesson we will continue looking at division, and the language we use to understand and solve problems. We will investigate division stories and consider how to deal with remainders. Finally, we will move on to learning about sharing leading to fractions. This is an important aspect of division, and one that needs much practice.

What you will learn in this lesson

• Division stories
• Division with remainders
• Sharing leading to fractions

Division stories

As learners develop their understanding of division, they will become more confident in recognising the difference between grouping and sharing. By providing learners with a variety of division stories, we will enable them to construct their own understanding of the strategies used to solve division problems.

It is important to encourage learners to read problems carefully in order to work out what the best strategy would be to solve the problem. By identifying the key parts of the division story, learners can then determine if they know how many groups they have or if they know how many objects go in each group. We need to present problems that involve the two different strategies. Think about these problems- learners need to be able to identify information they present in order to identify the strategy needed to solve it:

<table>
<thead>
<tr>
<th>There are 72 biscuits.</th>
<th>There are 85 flowers.</th>
</tr>
</thead>
<tbody>
<tr>
<td>I put 4 biscuits in a box.</td>
<td>There are 5 friends.</td>
</tr>
<tr>
<td>How many boxes do I need?</td>
<td>How many flowers should each friend get?</td>
</tr>
</tbody>
</table>

In the first problem, about the biscuits, the following information is given:

• The total number of biscuits
• The number of objects (biscuits) that will go in each group (box)
• The learners need to find out how many groups (boxes) they will need.
• This is a **grouping problem**.

However, in the problem with the flowers, the learners know:

• The total number of flowers
• The number of groups (friends)
• The learners need to find out how many objects (flowers) will go in each group.
• This is a **sharing problem**.
ACTIVITY 1
Make up a story that would lead to a number sentence involving division.

Commentary
As we saw when we discussed multiplication stories, it is important for the learners to be involved in problems that provide opportunities for success. They need to be able to develop their confidence by completing problems that have a similar structure and layout. You need to give problems that require grouping and sharing as solution strategies. Encourage the learners to create their own division stories, thinking about how to use the mathematical language that they have learnt. Make sure that you have modelled the appropriate use of the language so that the learners have a good example to follow. Remember to get learners thinking about whether their problems are sharing problems or grouping problems and ask them to explain their reasoning.

Division with remainders
The underlying concept of division is that we need break down quantities into equal sized groups. Once learners understand this, then remainders do not pose a threat. Learners will realise that remainders are a normal part of life, and they can discuss what they think should happen to the “left over” items (the remainder). Think about the following problem. How would learners solve it?

A farmer has 8 sheep pens, and 79 sheep.
During the day, the sheep graze in the veld
and at night they are put into the pens.
How many sheep will go into each pen?

This is a sharing problem, since the sheep have to be distributed into the pens. This can be done in the following way, allocating sheep to the pens evenly.

But what learners will find, is some sheep are ‘left over’. They will find that they can have 9 sheep in each pen, but then there will be 7 sheep left over. They will need to discuss what they think should happen to the remaining 7 sheep. Should they be left out in the veld on their own? Should one extra sheep be put in each pen so that 7 pens have 10 sheep and 1 pen has 9 sheep? Encourage learners to justify their ideas. In the problem:
Here is another problem learners could solve:

The farmer has 99 eggs.
He packs them for market in boxes of 12.
How many boxes did he fill?

This is a grouping problem, since the eggs have to packed in groups of a given number. This could be solved using repeated subtraction (or multiples of 10s and 2s).

The learners will realise that they will be able to pack 8 boxes of 12 eggs, and 3 eggs will be left over. They can discuss what they think they should do with the three eggs. Should they put the 3 eggs in a box on their own? Should they try to squash them into the boxes of 12? Or should they just eat them for breakfast?

**ACTIVITY 2**

Division with remainders in context

Watch the video “Division with remainders in context” (3:05 minutes), to find out how to deal with remainders.

- What did you notice about the learners’ answer to the problem?
- What did you notice about the teacher’s response?
Commentary

In the video, the teacher poses a real-life problem, and asks the learners to help her solve it. She encourages the learners to discuss how they would solve the problem, and to give an answer without using their blocks. The reason the teacher didn’t want the learners to use their blocks was firstly so that they could think logically about the problem. If the learners had used blocks, then they would have just said the answer is 4 with 3 left over. They wouldn’t necessarily have thought about the fact that the ‘3 left over’ were actually learners who would have nowhere to sit. It is important for learners to understand exactly what the question was asking, and to provide an answer to that question, rather than just the numerical facts.

Secondly, the teacher wanted the learners to see how she used the table on the board to help her solve the problem. The teacher shows the learners that they can count in multiples of six to find out how many benches will be needed to hold all 27 learners. The top row shows the number of benches, and the bottom row goes up in multiples of 6. This creates a visual representation that helps learners to see how they should answer the question.

Sharing leading to fractions

Many people find the concept of fractions a bit tricky, and this can result in some maths anxiety for both teachers and learners. However, it is important to note that if you have allowed learners to develop a solid understanding of division using concrete apparatus, this leads well into the learning about fractions.

Up until now, we have discussed division in terms of breaking quantities into equal groups. As we start looking at sharing leading to fractions, we can move towards dividing a shape into equal parts. As we can see from the images below, we can show learners that 1 rectangle can be divided into 2 equal parts.

This is a good time to remind learners that the fractional parts need to be the same size. Posing a problem about dividing a cake fairly provides a context for this discussion. Learners will realise that it would not be fair to cut a cake with slices cut in a variety of sizes. You can use the concept of fairness to stimulate learners to think about equal sized parts. Learners will be quick to point out that all the slices need to be the same size. In reality, it is difficult to cut circular shapes into exactly equal parts but you should try. A loaf of bread could also be used as an example. But paper folding can be done more easily in a class.

ACTIVITY 3

Think about how you can use paper folding to help learners grasp sharing leading to fractions.

• Why is paper folding a good why for learners to learn about sharing and fractions?
• Describe how you could use a paper folding activity to teach learners about breaking a whole into 4 parts.
Commentary

Paper folding is a useful way of demonstrating to learners that one whole (one paper shape) can be divided into equal parts. A benefit of paper folding is that it is very obvious that the parts are all equal sizes. These parts can be folded on top of each to show that they are equal, or they can be cut up and placed on top of each other. This activity emphasises for learners that fractional parts are exactly the same size, and it will give them the opportunity to use the correct mathematical language as they describe their parts. We will discuss more about fractions in the last 3 lessons of this part of the course.

Check your understanding: True or False?

1) Learners need to practice a variety of both grouping and sharing problems.

2) Remainders should not be taught until learners are older.

3) Division problems do not have to be related to real-life situations.

4) Paper folding should only be done with very young learners.

REFLECTION

• Reflect on your own views on remainders.
• Think about whether you felt anxious about dealing with reminders when you were at school.
• Write a personal goal in relation to how you will approach the teaching and learning of remainders in your classroom.

Well done you have completed Lesson 6.
In this lesson, we will investigate how to solve multiplication and division problems that involve higher numbers. This is often quite challenging for learners, so it is important that you as the teacher are comfortable with different methods of solving problems. In this way, you will be able to support learners as they construct their own understanding. In this lesson, we will also look at multiplication and division as inverse operations and spend time considering some common learner errors.

**What you will learn in this lesson**

- Multiplication and division with larger numbers
- Multiplication and division as inverse operations
- Learner errors

### Multiplication and division with larger numbers

Learners typically find it extremely difficult to multiply and divide large numbers. It is for this reason that we need to ensure that they have developed a sound understanding of number relationships through the use of concrete resources. Learners need a variety of problems so that they can practice a range of solution strategies once they move onto doing numeric calculations.

If learners are able to understand what they are doing when they multiply and divide, then they can find ways to solve problems without relying on learnt algorithms. For example, if learners have an understanding of the laws of multiplication, then they will be able to solve the problem $3 \times 28$ quite easily. Rather than trying to count in 28s, or trying to write out the multiplication algorithm in column format, they can break up the problem into:

$$20 \times 3 + 8 \times 3 = \square$$

$$60 + 24 = 84$$

This is far easier to solve in your head than the problem $3 \times 28$.

In the same way, with division, if learners are given the problem $462 \div 2 = \square$ then they can use their knowledge of partitioning to help them solve it more efficiently. Learners can break the number 462 into $400 + 60 + 2$, and then divide each part by 2 which can be done mentally.

$$400 \div 2 = 200$$

$$60 \div 2 = 30$$

$$2 \div 2 = 1$$

$$200 + 30 + 1 = 231.$$
ACTIVITY 1

Watch the video “Division of 2-digit numbers” (3:38 minutes), to see how the learners learn to divide 2-digit numbers.

- How did learners use the base ten blocks to help them?
- What other method could learners have used to solve the problem?

Commentary

The learners used their blocks to share out the 39 between 3 groups. In doing this, they could see that \( 39 \div 3 = 13 \) because each group got 1 ten and 3 ones. It was clear that this was a sharing problem, because the learners knew how many groups there were, but they didn’t know how many went into each group. Learners could have used partitioning to solve this problem in a different way. They could have broken down 39 into 30 + 9, and then divided each number by 3.

\[
\begin{align*}
30 \div 3 &= 10 \\
9 \div 3 &= 3 \\
10 + 3 &= 13
\end{align*}
\]

Multiplication and division as inverse operations

Division and multiplication are inverse operations. This means that division “undoes” what multiplication “does”. Where learners have had lots of exposure to times tables they will more easily see that division is the inverse of multiplication. You should encourage learners to build up their division tables alongside their times tables so that they can identify the patterns for themselves.

Give the learners practical examples to solve, letting them think about how they can work out the problems. For example:

<table>
<thead>
<tr>
<th>I have 42 pencils.</th>
<th>42</th>
</tr>
</thead>
<tbody>
<tr>
<td>I share the pencils between 6 friends.</td>
<td>6</td>
</tr>
<tr>
<td>How many pencils will each friend get?</td>
<td>?</td>
</tr>
</tbody>
</table>

If learners think about a part-part-whole model, they will realise that they can solve the problem by simply recalling their 6 times table. It is critical that learners reach the stage where they can use multiplication facts to solve division problems.
ACTIVITY 2
Give learners a division problem to solve.
• Watch learners solve the problem and see their level of understanding.
• Ask learners to use multiplication to check their solution and see whether they are able to use the inverse operation to clarify their understanding.

Commentary
The learners will need to remember that they need to obey the laws of multiplication and division. This means that the numbers in the number sentences will have to be written in a certain order in order to abide by these rules. Learners can find this tricky, and this is why it is useful to use a part-part-whole diagram to help them remember which number is the ‘whole’ and which number is a ‘part’. For example, if the learners were given the problem:

<table>
<thead>
<tr>
<th>There are 96 flowers.</th>
</tr>
</thead>
<tbody>
<tr>
<td>They are put into 8 vases.</td>
</tr>
<tr>
<td>How many flowers must go into each vase?</td>
</tr>
</tbody>
</table>

By using a part-part-whole diagram, the learners will most probably recognise that the problem can be solved by using their knowledge of the 8 times table. They can use the diagram to identify the following number facts:

\[
\begin{align*}
96 \div 8 &= 12 \\
96 \div 12 &= 8
\end{align*}
\]

Learner errors
Teachers need to anticipate possible areas where misconceptions may arise, and division is one topic where some confusion may develop. For example, when teaching learners about sharing leading towards fractions, learners tend to make a common error where they say the number in a group instead of the number of groups. If you anticipate this mistake, then it is easier to address it and re-direct understanding during your teaching, rather than trying to undo learnt ideas. Misconceptions cannot simply be uprooted and replaced with new, “correct” concepts. And they can be difficult to change.

Some learners find it difficult to let go of oversimplified rules that they have learned in the early grades, especially if these seem to be simple and clear. Some misconceptions originate in teachers’ efforts to make content and procedures simple for learners in the early years of schooling. Teachers need to find out what learners have been taught and where necessary to show them what is inadequate or incorrect about what they have previously learned.

ACTIVITY 3
A learner in your Grade 3 class made the following mistake:

\[
\frac{1}{3} + \frac{1}{2} = \frac{3+1}{3+2} = \frac{2}{5}
\]

• Explain the mathematical error that the learner has made.
• What explanation would you give to help the learner to rectify the error?
Commentary

The learner has wrongly applied the rule “what you do to the top you do the bottom” and arrived at the incorrect answer. To assist learners to rectify this error the teacher could use concrete materials to demonstrate addition of fractions so that learners see that half a cake (or an apple) plus a third of a cake or apple cannot be two fifths of a cake or apple. The concrete demonstration will make two things clear to the learner. The first is that unlike fraction parts cannot be added (e.g. $\frac{1}{3} + \frac{1}{2}$ cannot be added as they are). The second is that like fractions can be added by adding the number of fraction parts (e.g. $\frac{1}{3} + \frac{1}{2} = \frac{2}{6} + \frac{3}{6} = \frac{5}{6}$).

Check your understanding: Multiple Choice

1) All learners should:
   A) Solve problems in the same way.
   B) Be given opportunities to try different solution strategies.
   C) Be taught one way to solve problems.

2) Partitioning is:
   A) A more time-consuming way to solve problems.
   B) Only used to demonstrate place value.
   C) An efficient method that can be done mentally.

3) Inverse operations:
   A) Can be used to check answers to problems.
   B) Do not need to be taught.
   C) Should only be taught to older learners.

4) Commonly used phrases in the classroom can be:
   A) A way to help learners remember what to do.
   B) Quick and easy signposts while teaching.
   C) A way of creating misconceptions.

REFLECTION

• Reflect on your use of mathematical language in the classroom.
• Think about whether you are creating misconceptions by the way you say things to the learners.
• Write a personal goal in relation to how you plan to monitor what you say in the classroom.

Well done you have completed Lesson 7.
Fractions are numbers, and the development of fraction number concept in the early grades lays the foundation for further teaching about rational numbers in later years. Rational number concept involves an understanding of fractions which involves more than just the finding of parts of a whole. Learners need to be exposed to a range of activities and conceptual teaching on fractions as parts of wholes, ratios, decimals and percentages in order to develop fully their understanding of multiplicative reasoning and rational numbers. Much of this learning happens beyond the Foundation Phase but it is important to see the teaching of concepts in the bigger context.

In the next three lessons you will learn about the teaching of fractions. You will be taken through a series of activities that show you how to provide learners with opportunities to work with a wide variety of wholes (continuous and discontinuous) to enable the development of a solid concept of fractions as numbers.

What you will learn in this lesson

- Fractions as parts of wholes – introducing fractions
- The difference between continuous and discontinuous wholes
- Language to use when finding fractions as parts of wholes

Fractions and wholes

Fractions can be used to represent numbers which are not whole numbers. As such, they are slightly more difficult to come to terms with than whole numbers and are taught once basic number concept has been established. The first part of this activity will look in a detailed manner at sound methods for the teaching of fractions to young learners. You should be able to follow these ideas and ensure that all of the information given forms part of your own knowledge. It is vital that all teachers of mathematics have a good concept of fractions themselves.

We need to ensure that learners are given adequate exposure to a great enough variety of examples of fractions in concrete demonstrations so that they are able to form their own abstract concept of what number the fraction numeral represents. We begin by looking at fractions as parts of concrete wholes and progress from there to more abstract working with fractions.

Types of wholes

The first important thing we should stress is that we can find fractions of continuous and discontinuous wholes. These two types of wholes are not always given equal representation in workbooks and activities. We should not emphasise one more than the other or we risk giving an unbalanced idea of concrete wholes.
Lesson 8

**Continuous whole**

- single items that make up the whole also called unit wholes
- Fraction parts are equal sized parts into which the unit whole has been divided
- Examples: an orange, a piece of paper, a slab of chocolate, a circular disc, a loaf of bread

**Discontinuous whole**

- groups of items that together make up the whole
- Fraction parts are equal sized groups (same number of items in each group)
- Examples: 15 oranges, 6 biscuits, 27 counters, 4 new pencils

**ACTIVITY 1**

- List and describe five of your own examples of continuous wholes.
- List and describe of your own examples of discontinuous wholes.

**Commentary**

When deciding on your own examples, remind yourself of the key features of each type of whole. The continuous wholes should be **one single item** that can be cut/broken up to find the fraction part, and the discontinuous wholes should be **multiple item wholes** (anything more than one) that have to be grouped to find the fraction part. Learners need to be exposed to as many different examples of wholes when working with concrete items or diagrammatic representations in order to be able to develop a well generalised fraction number concept.

**Language to use when finding fractions as parts of wholes**

To assist learners to establish their fraction concept, we must use good language patterns consistently when talking about fractions. It is thought that our language is linked to our thinking, and so by encouraging learners to talk about what they see, we help learners to transfer what they see in the concrete demonstrations into their abstract thought. The language patterns that we are talking about link to the different kinds of wholes that learners encounter when working with concrete (real objects) and semi concrete (drawings of objects) examples.
Language patterns (talking about) continuous wholes

It is essential to use the correct language patterns when teaching learners about fractions. This can be a tricky concept for learners, and they often have misconceptions about the terminology of fractions. For example, when sharing a piece of cake, one learner may complain that “My half is smaller than your half!” This incorrect use of the word half shows that the learner does not understand that fractional parts need to be exactly the same size. Below is an example to show you what it means to use correct language when working with a continuous (single unit) whole.

ACTIVITY 2

Watch the video "Unitary fractions of continuous wholes – teacher" (4:48 minutes), to see how the teacher demonstrates finding fraction parts.

After watching the video try out the following activities for yourself:

• Find \( \frac{1}{3} \) of the rectangle given below:

• Shade \( \frac{1}{4} \) of a strip of paper.
• Illustrate and explain how to find \( \frac{1}{6} \) of a circular cake.

Commentary

Practice the correct language pattern to use while speaking aloud about finding fraction parts. The use of good language allows learners to develop thinking patterns since it gives learners verbalisation skills that can support abstract thinking. This activity focuses on language patterns when speaking about fraction parts of continuous (single item) wholes.
• To find \( \frac{1}{3} \) of the rectangle, I first divide the whole rectangle into 3 parts of equal size. Each part is \( \frac{1}{3} \) of the whole, and if I shade one of these parts, I have shaded \( \frac{1}{3} \) of the whole rectangle.

• To find \( \frac{1}{4} \) of my paper strip, I first divide the strip into 4 parts of equal size. Each part is \( \frac{1}{4} \) of the whole, and if I shade one of these parts, I have shaded \( \frac{1}{4} \) of the strip.

• To find \( \frac{1}{6} \) of my circular cake, I first divide the whole circular cake into 6 parts of equal size. Each part is \( \frac{1}{6} \) of the whole, and if I shade one of these parts, I have shaded \( \frac{1}{6} \) of the circular cake.

Language patterns (talking about): discontinuous wholes

Here is an example to show you what it means to use correct language when working with a discontinuous (multiple unit) whole.

Find \( \frac{1}{8} \) of 32 counters

32 counters (shown above) represent the whole.

I put my counters into 8 groups of equal size. There are four counters in each group.

One of the groups of equal size is \( \frac{1}{8} \) of the whole.
ACTIVITY 3

Unitary fractions of discontinuous wholes – teacher

Watch the video “Unitary fractions of discontinuous wholes – teacher” (3:36 minutes), to see how the teacher demonstrates finding \( \frac{1}{3} \) of 12.

After watching the video try out the following activities for yourself:

• Find \( \frac{1}{3} \) of 27 oranges, as given below.

![Image of 27 oranges divided into three groups of 9 each]

• Find \( \frac{1}{3} \) of 30 marbles.

• Find \( \frac{1}{4} \) of 20 cupcakes.

Commentary

This activity focuses on language patterns when speaking about fraction parts of discontinuous (multiple item) wholes. Notice how with a discontinuous whole you work with groups of items according to the fraction you have to find. In the first activity the oranges have been drawn. Learners should draw dots (or other representations) if they need a drawing to help them find the fraction parts.

• To find \( \frac{1}{3} \) of 27 oranges, I first divide the oranges into 3 groups of equal size. I find three groups with nine oranges in each group. Each group is \( \frac{1}{3} \) of the whole, and so 9 oranges is \( \frac{1}{3} \) of 27 oranges.

![Image of 27 oranges divided into three groups of 9 each]

• To find \( \frac{1}{3} \) of 30 marbles, I first divide the marbles into 3 groups of equal size. I find three groups with 10 marbles in each group. Each group is \( \frac{1}{3} \) of the whole, and so 10 marbles is \( \frac{1}{3} \) of 30 marbles.

![Image of 30 marbles divided into three groups of 10 each]
• To find $\frac{1}{4}$ of 20 cupcakes, I first divide the cupcakes into 4 groups of equal size. I find four groups with five cupcakes in each group. Each group is $\frac{1}{4}$ of the whole, and so 5 cupcakes is $\frac{1}{4}$ of 20 cupcakes.

Check your understanding: True or False?

1) A packet of biscuits is an example of a continuous whole.
2) A circular disc is an example of a continuous whole.
3) 25 beads is an example of a discontinuous whole.
4) It does not matter what the whole is when you look for fraction parts.

REFLECTION

• The concept of a fraction grows from a concrete understanding of a part of a whole to an abstract understanding of a fraction as a number. Reflect on what this means in relation to your teaching of fractions.

• When you introduce fractions to learners do you give them a lot of opportunities to work with concrete manipulatives? Describe some of the activities you do.

Well done you have completed Lesson 8.
Curriculum coverage is never just about completing the learning programme designed by a teacher or school. In this series of lessons on fractions all of the curriculum content of the early grades as well as an extension of some of these concepts is presented. In this lesson we focus on fractions as numbers.

What you will learn in this lesson
• How foundational fraction number concept fits into the development of number concept
• Unit fractions and non-unit fractions
• More about continuous and discontinuous wholes

Teaching fractions as part of the curriculum

Teachers need to think carefully about the levels of understanding of content that are appropriate to the age-level of the learners. This means that it is important to think about the relation between content and concepts. The content of a topic consists of concepts, which organise the information. Therefore, concepts need to be taught (or learned) in a particular order since they are made up of different levels of complexity.

Concepts also regulate what can and should be connected with what, and they give a form or a shape to the content. Concepts give a shape to the information by showing which piece of information is more central than another, which should come before the other, which is bigger and which is smaller etc. In order to know content well, teachers and learners need to understand the concepts that inform or shape it. Fraction number concept develops over time and can be summarised in the following concept map.

What the concept map about fractional thinking shows is that concepts structure the content into a network of ideas.
ACTIVITY 1

How should a learner answer this question:

Which of the following circles has NOT been shaded in halves?

A B C D

- What is the correct answer?
- What conceptual knowledge does the learner need in order to get to the correct answer?
- Does this conceptual knowledge ‘fit’ into the concept map above? If so, where?

Commentary

To get the correct answer of B, learners need to have an understanding that there are many different ways of showing a half of a whole in a concrete representation. They needed to be alert when reading the question to realise that they had to identify which image did NOT represent a half. Recognising a half of a given circular disc involves recognising that the whole must be divided into two parts of equal size. The half can be made of smaller pieces (such as two quarters) that do make up a half when taken together. This means that it would fit into the concept map above under ‘Introduction – fraction as a part of a whole’.

Unit fractions and non-unit fractions

When you introduce fractions to learners, you will begin by finding unit fractions (as we have done above). A unit fraction is a fraction of the form \( \frac{1}{n} \). The numerator is one and the denominator can be any number except zero. You must allow the learners to experiment with finding unit fractions of a broad variety of wholes.

At the beginning you will restrict your discontinuous wholes according to the denominator. For example, if the denominator is 6, you will only ask the learners to find fraction parts of 6 counters, or 12, 18, 24, etc. counters (multiples of 6). You must also remember to set tasks involving continuous wholes as well as discontinuous wholes.

Vary your apparatus as widely as you can. Use pieces of paper, string, sand, water, beads, counters, strips of paper, bottle tops – whatever is easily available.

Continuous whole example of a unitary fraction

\[ \frac{1}{4} \] of 1 square = \( 1 \div 4 = \frac{1}{4} \) of the square. (In the continuous whole example the answer is a fraction, not a whole number.)

Discontinuous whole example of a unitary fraction

\[ \frac{1}{4} \] of 20 stamps is \( 20 \div 4 = 5 \) stamps. (In the discontinuous whole example the answer is a whole number.)

Once you are satisfied that your learners have established the general result: \( \frac{1}{n} \) of \( m = m \div n \), you can move on to finding non-unit fractions (these will be discussed later in the session).

You will now set tasks for your learners to find fraction parts of wholes, where the fraction is of the type \( \frac{m}{n} \) where \( n \neq 0 \). This is purely an extension of the previous activities, where you found \( \frac{1}{n} \) of a whole. These fractions are called non-unitary fractions. Learners should not experience too many difficulties finding non-unitary fractions if unit fractions have been grasped well. Examples of finding non-unitary fractions of a continuous and then a discontinuous whole are shown below.
**Continuous whole: Find \(\frac{5}{6}\) of a square sheet of paper.**

<table>
<thead>
<tr>
<th>The whole.</th>
<th>The whole divided up into 6 parts of equal size.</th>
<th>5 of the 6 parts of equal size have been shaded. (\frac{5}{6}) of the whole has been shaded.</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image1" alt="Whole" /></td>
<td><img src="image2" alt="Whole divided into 6 parts" /></td>
<td><img src="image3" alt="5 parts shaded" /></td>
</tr>
</tbody>
</table>

**Language pattern**

The whole is a square sheet of paper. I fold the whole up into six parts of equal size in order to find sixths. Each part is \(\frac{1}{6}\) of the whole, so 5 of the six equal sized parts is \(\frac{5}{6}\) of the whole.

---

**ACTIVITY 2**

**Non-unitary fractions of continuous wholes – teacher**

Watch the video “Non-unitary fractions of continuous wholes – teacher” (4:52 minutes), to learn more about non-unitary fractions of continuous wholes.

- Illustrate \(\frac{2}{3}\) of a pizza.
- Shade \(\frac{3}{5}\) of a rectangular sheet of paper.

**Commentary**

When you do these activities, make an effort to use the full, correct language when you speak about finding fraction parts of the various given wholes. Notice how with a continuous whole you cut/break/divide the whole according to the fraction you have to find. Since you must find a non-unit fraction, you must take more than one of the equal parts into which you divided the whole.

- To find \(\frac{2}{3}\) of my pizza, I first divide the whole pizza into 3 parts of equal size. Each part is \(\frac{1}{3}\) of the pizza, and if I shade two of these parts, I have shaded \(\frac{2}{3}\) of the pizza.
- To find \(\frac{3}{5}\) of my rectangular sheet, I first divide the whole rectangular sheet into 5 parts of equal size. Each part is \(\frac{1}{5}\) of the rectangular sheet, and if I shade three of these parts, I have shaded \(\frac{3}{5}\) of the rectangular sheet.
Discontinuous whole: Find $\frac{3}{4}$ of 36 marbles

The whole.

The whole divided into quarters.

Three of the four groups (representing $\frac{3}{4}$ of 36) have been shaded.

Language pattern
The whole is 36 marbles. I divide the whole up into four groups of equal size in order to find quarters. There are 9 marbles in each group. One group of 9 is $\frac{1}{4}$ of 36, and so 3 groups of 9 are $\frac{3}{4}$ of 36, so 27 is $\frac{3}{4}$ of 36.

ACTIVITY 3

Non-unitary fractions of discontinuous wholes – teacher

Watch the video "Non-unitary fractions of discontinuous wholes – teacher" (5:40 minutes), to find out more about non-unitary fractions of discontinuous wholes.

After watching the video try out the following activities and write out the full language pattern you would use in each case, so that you can check your own ability to talk fluently about the fraction parts you are finding and model it for learners.

• Show how you find $\frac{2}{5}$ of 20 bricks.
• What is $\frac{3}{4}$ of 20 liquorice strips?
Commentary

When you do these activities, make an effort to use the full, correct language when you speak about finding fraction parts of the various given wholes. Notice how with a discontinuous whole you work with groups of items according to the fraction you have to find. Since you must find a non-unit fraction, you must take more than one of the equal groups into which you divided the whole.

- To find $\frac{2}{5}$ of 20 bricks, I first divide the bricks into 5 groups of equal size. I find five groups with 4 bricks in each group. Each group of 4 is $\frac{1}{5}$ of the whole, and so 8 bricks is $\frac{2}{5}$ of 20 bricks.

- To find $\frac{3}{4}$ of 20 liquorice strips, I first divide the liquorice strips into 4 groups of equal size. I find four groups with five liquorice strips in each group. Each group of 5 is $\frac{1}{4}$ of the whole, and so 15 liquorice strips is $\frac{3}{4}$ of 20 liquorice strips.

You could turn some of your fraction finding into games or activities. In this way, you could keep the learners busy for slightly longer periods of time, while they are learning and discovering ideas in an interesting and enjoyable way. In this example, learners are given 20 counters. They must then try to find all the possible fraction parts that they can, of 20 counters. They could work in groups of two to four members (not more, as they would not have enough of a chance to express themselves). The discussion of the different fraction parts could go on in the whole group. Once the group thinks that they have found all the possible fraction parts they can put up their hands and say “Full House!”, to call you to come and check up on them. As a follow up, ask each learner to record in full and good language one of the fraction parts which they found. Try this activity out yourself!

Check your understanding: True or False?

1) $\frac{3}{4}$ is a non-unitary fraction.
2) A piece of paper is an example of a discontinuous whole of which you could find a half.
3) $\frac{3}{4}$ of 24 apples is 6 apples.
4) You can use examples of incorrectly shared fractions when you teach learners about fractions.

REFLECTION

- What is the benefit of giving learners many varied examples of whole when teaching them about fractions?
- What is the benefit of giving learners many varied examples of fractions to find when teaching them about fractions?
- Write a personal goal about the use of manipulatives when teaching fractions in your class.

Well done you have completed Lesson 9.

<table>
<thead>
<tr>
<th>Check your understanding: True or False?</th>
</tr>
</thead>
<tbody>
<tr>
<td>1) $\frac{3}{4}$ is a non-unitary fraction.</td>
</tr>
<tr>
<td>2) A piece of paper is an example of a discontinuous whole of which you could find a half.</td>
</tr>
<tr>
<td>3) $\frac{3}{4}$ of 24 apples is 6 apples.</td>
</tr>
<tr>
<td>4) You can use examples of incorrectly shared fractions when you teach learners about fractions.</td>
</tr>
</tbody>
</table>

Answers:

1) True
2) False. A piece of paper is an example of a continuous whole, but you can find a half of the piece of paper by folding it into two equal sized parts.
3) False. 6 is one quarter of 24, $\frac{1}{4}$ of 24 apples is $\frac{3}{4} \times 6 = 18$ apples.
4) True. You can use such examples to give learners an opportunity to reflect what you show them as a correct shading. This helps you check their understanding.
In the early grades, learners are not expected to learn a lot of fraction terminology, but as a teacher of maths you need to know more than you are expected to teach so that you could answer questions asked by learners and go beyond the basic curriculum expectations. In this lesson some important fraction terminology is introduced to develop your knowledge of fractions. There is also input on equivalent fractions and comparing fractions. Finally, you will be given the opportunity to think about dealing with learner misconceptions about fractions.

**What you will learn in this lesson**

- Fraction numerals, like and unlike fractions, proper and improper fractions
- Equivalent fractions
- Comparing fractions
- Working with fractions misconceptions

**Fraction numerals**

Show learners how to write a fraction numeral and tell them the terminology. Make sure they know which of the numerals is the numerator (the number at the top of the fraction numeral) and which is the denominator (the number at the bottom of the fraction numeral).

5 \[\underline{\text{numerator}}\]

6 \[\underline{\text{denominator}}\]

You must learn these names if you do not already know them. This is important terminology in the section of fractions. Make sure that learners use the terminology repeatedly, to help them build the words into their regular speech.

**Like and unlike fractions:**

We call fractions which have the same denominators like fractions. Fractions whose denominators are not the same are called unlike fractions. For example \(\frac{3}{7}, \frac{6}{7}\), have 7 as their denominator.

**Proper and improper fractions**

When the numerator of a fraction is smaller than the denominator of a fraction, the fraction is called a proper fraction. When the numerator of a fraction is bigger than the denominator of a fraction, the fraction is called an improper fraction.
Equivalent fractions

Your learners will already have begun to notice certain equivalent fractions before you consciously introduce the topic in class. They might have begun to say things to you like “but $\frac{2}{4}$ is the same as $\frac{1}{2}$.” You should encourage this early observation even if it is not called for in curriculum specifications. You could possibly even comment that they have noticed an important quality that they will learn more about later. Here is an activity that you could use to help clarify the understanding of equivalent fractions using concrete wholes.

### ACTIVITY 1

Take 5 pieces of paper that are the same size. Fold each of them into thirds using vertical folds, as illustrated below. Shade in the first third on each piece of paper.

![Illustration of folded papers]

Now fold pieces B, C, D and E using horizontal folds as indicated below:

![Illustration of folded papers]

What fraction of each piece of paper has been shaded?

### Commentary

The fraction represented by the shaded part on each piece of paper is the following:

- A $\frac{1}{3}$ is shaded
- B $\frac{2}{6}$ is shaded
- C $\frac{1}{3}$ is shaded
- D $\frac{4}{12}$ is shaded
- E $\frac{5}{15}$ is shaded

All of these fractions have the same value (one third or $\frac{1}{3}$) although they are written in different ways with different fraction numerals and spoken about using different fraction names (for example two sixths or $\frac{2}{6}$, and so on).

### Comparing fractions

It is natural to compare whether or not certain numbers represent more or less than other numbers. When we do so for fractions, this process is sometimes fairly involved. In early grades, comparisons of fractions are based on concrete representations, which lays the foundation for work in later grades with purely numeric examples. When we compare fractions, we often ask or need to find out “which one is greater and by how much?” The solution to this is not always as clearly evident as it is in whole number questions. Here is an example of such a comparison.
ACTIVITY 2

Let’s compare $\frac{2}{3}$ and $\frac{3}{4}$, using a concrete discontinuous whole. Let 12 flowers be our whole.

The whole

$\frac{2}{3}$ of 12 is 8

$\frac{3}{4}$ of 12 is 9

Which fraction is greater and by how much?

Commentary

You should do this activity with concrete material yourself – use counters to represent flowers, as drawings are static and much less convincing (or satisfying) when doing introductory examples. To work out which fraction is greater we need to look at the number of flowers we found for each fraction part. If you do this concretely, you should find that $\frac{3}{4}$ is greater than $\frac{2}{3}$ by one flower.

But what is the value of one flower? If there are 12 flowers in the whole, then one flower is $\frac{1}{12}$ of the whole. We can thus say that $\frac{3}{4}$ is greater than $\frac{2}{3}$ by $\frac{1}{12}$.

Fraction walls can also be used to compare the sizes of different fraction parts. You could show your learners this fraction wall and ask them the following questions. They should answer with reference to the fraction wall to support their reasoning about the relative sizes of given pairs of fractions.

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Misconceptions and fractions

Misconceptions are part of the knowledge that learners develop, and so form part of their current knowledge. Teachers need to build on learners’ current knowledge. This means that they need to listen to, work with and build on learners’ misconceptions as well as on their correct conceptions. Learning always involves transforming current knowledge. Therefore, if misconceptions exist, the existing knowledge will need to be transformed and re-structured into new knowledge. If teachers listen to and work with learners’ thinking they can learn about:
• the types of conceptions learners have;
• how to build onto these conceptions;
• how to use misconceptions to transform learners’ thinking and to inform teaching.

Here is an activity that gives you an opportunity to think about learners' misconceptions in relation to fraction concept and how to work with these misconceptions in a constructive way. This will help learners come to correct conceptualisation of the concepts.

**ACTIVITY 3**

This multiple-choice question about fractions (and the way learners answered it) gives insight into learners’ thinking about fraction number concept.

Tshepo has a slab of chocolate with 32 pieces. He broke off a quarter of the slab. Which of these drawings shows a quarter of the slab of chocolate?

A B C D

• First think about the correct answer to the question yourself.
• In a sample of learners that answered the question in a test they had, 25% chose the correct answer (A) while 31% chose the incorrect answer (B). There were also learners that chose the other two incorrect options.
• What does this show you about the way learners understand pictorial representations of fractions?

**Commentary**

The correct answer to this question is A, because in this case the number of blocks in the whole (32) has been divided into four equal parts of 8. 8 blocks is a quarter of 32 blocks.

Answer B – the most highly chosen (incorrect) answer, was probably chosen because it "looks" most like the fraction parts into which learners have most often seen shapes being divided. The other shapes, given as potential quarters have unusual, irregular shapes. B was probably chosen because the learners saw it as a more familiar shape (a rectangle). This means they did not count the individual blocks since there are only 6 blocks in this rectangle which is not one quarter of 32. Learners are often not exposed to fraction parts which are irregular in shape. The familiarity which might have influenced their choice here disadvantaged the learners.

The misconception seen here is called an over-generalisation. Learners probably chose it because it is the most familiar type of “quarter” of the four options, based on the generalisation that when a whole is divided into quarters they will look like this. They did not evaluate the size of the given part (six pieces) in relation to the whole (which was said to have 32 pieces). This results when learners have not been shown enough of a variety of different ways in which wholes can be broken up into parts.
Check your understanding: Multiple Choice

1) An improper fraction is:
   A) When the numerator of a fraction is smaller than the denominator of a fraction.
   B) When the numerator of a fraction is bigger than the denominator of a fraction.
   C) When the numerator of a fraction is the same size as the denominator of a fraction.

2) Equivalent fractions should be:
   A) Discussed naturally as learners ask questions.
   B) Avoided at all costs.
   C) Only covered at the appropriate time in the curriculum.

3) Learning should:
   A) Happen in such a way that misconceptions are impossible.
   B) Ignore minor misconceptions.
   C) Always involve transforming current knowledge.

4) Over-generalisations occur when:
   A) Learners have not been shown enough of a variety of methods.
   B) Learners have been shown too many different methods.
   C) Learners are guessing at the answers.

REFLECTION

• Reflect on why it is important for you as a teacher to have knowledge that goes beyond what you have to teach your early grade learners.

• Write a personal goal in relation to ways in which you could build your own knowledge of fractions so that you can give learners the best possible learning experience in your classroom.

Well done you have completed Lesson 10.